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THE INFLUENCE OF AXIAL FORCES ON THE COLLAPSE LOADS OF FRAMES

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The Influence of Axial Forces on the Collapse Loads of Frames

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Abstract. In the limit analysis of frames the influence of the axial forces is usually neglected because the limit moment of a section is not reduced seriously by a moderate axial force. In an earlier paper concerned with the limit analysis of arches[3] it was shown that the presence of axial forces requires the use of yield hinges that allow not only relative rotation but also relative axial displacement of adjacent cross sections. In the present paper a different approach is discussed: the extensible hinges on the center line are replaced by ordinary hinges off the center line. It is shown that, under certain assumptions, the location and the limit moment of an off-center hinge are independent of the axial force. The collapse load of a frame in which axial forces must be considered is thus found to equal the collapse load of a slightly modified frame in which axial forces may be neglected.

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3 E. T. Onat and W. Prager, "Limit analysis of arches", J. Mech. Phys. Solids 1, 77 (1953).

The concepts of limit analysis were developed in the twenties in connection with the design of continuous beams[4]. The application of limit analysis to the design of frames constitutes a relatively recent advance[5]. There are several reasons for this surprising time lag in the development of neighboring fields of structural engineering. Firstly, even for continuous beams experimental corroboration of the basic hypotheses was slow in forthcoming[6]. Secondly, the simple techniques that were adequate for continuous beams did not prove useful for complex frames; only in the last few years have efficient techniques been developed[7]. Finally, there were the difficulties created by the presence of axial forces in frames. In addition to raising the question of structural stability, axial forces lower the limit moments of the members in which they act.

As regards stability, both limit analysis and elastic analysis are based on the assumption that all compression members

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4 M. C. Kist, "Die Zaehigkeit des Materials als Grundlage fuer die Berechnung von Bruecken, Hochbauten, und aehnlichen Konstruktionen aus Flusseisen", Eisenbau 11, 425 (1920).

5 J. F. Baker, "A review of recent investigations into the behaviour of steel frames in the plastic range", J. Inst. C.E. 31, 188 (1949).

6 H. Maier-Leibnitz, "Test results, their interpretation and application", Prelim. Publ., 2nd Congress, Internat. Assoc. Bridge Struct. Engg., Berlin, 1936, p. 97.

7 P. S. Symonds and B. G. Neal, "Recent progress in the plastic method of structural analysis", J. Franklin Inst., 252, 383, 469 (1951).

have sufficiently low slenderness ratios to ensure structural stability. Only after the axial forces have been determined under this assumption can the possibility of buckling be investigated. Of course, the results of the first analysis are accepted only if the subsequent check shows that there is no danger of instability. Otherwise, the critical members must be strengthened, and the analysis must be repeated.

The influence of the axial forces on the limit moments can be treated by a similar iterative procedure. In the first step of the analysis the limit moments in the absence of axial forces are used. The axial forces found in the first step are then used to determine reduced limit moments for the second step, and so on. Two or three steps are usually sufficient. Whereas this method is suited for the numerical solution of concrete problems, it does not yield general results.

Instead of retaining the concept of the yield hinge and discussing the influence of an axial force on the limit moment in this hinge, consider the manner in which this concept must be modified for members carrying bending and direct stresses. For brevity, the following discussion will be restricted to members that are symmetric with respect to the plane of bending. Moreover, all elastic deformations will be neglected. This amounts to treating the perfectly plastic material as rigid whereever the stresses are below the yield limit.

The state of stress at a generic cross section is specified by the bending moment M and the axial force N. The generalized strain rates corresponding to the generalized stresses

M, N are the rate of bending ψ and the rate of extension ε. The relative angular velocity of two neighboring cross sections is ψ dx, where dx is the infinitesimal distance between these sections, and the relative axial velocity of their centroids is ε dx. The sign conventions for ψ and ε are so adjusted to those for M and N that

$$D\,dx = (M\psi + N\varepsilon)dx \qquad (1)$$

represents the rate at which mechanical energy is dissipated in the plastic flow of the material between the two cross sections. The quantity $D = M\psi + N\varepsilon$ will be called the specific rate of dissipation.

The neutral axis, which is normal to the plane of bending, divides the section into two parts. If the plastic-rigid member is to deform at all, the axial stress must equal the yield limit in tension at all points in one part and the yield limit in compression at all points in the other part. With this type of stress distribution, each assumed position of the neutral axis furnishes a combination of M and N that will produce plastic flow. Moreover, under the usual assumption that material cross sections remain plane during the bending process, the position of the neutral axis specifies the ratio between ψ and ε.

Figure 1 shows a symmetric section with the centroid C, and two fully plastic stress distributions corresponding to neighboring positions of the neutral axis. The compressive stresses which must be added to the first distribution to obtain the second one are represented by the shaded rectangle. The corresponding changes in bending moment and axial force are

$$dM = 2\sigma_o\, by, \qquad dN = -2\sigma_o\, b, \tag{2}$$

where σ_o is the intensity of the yield stress in tension or compression, y the distance of the neutral axis from the centroid, and b the width of the section at the distance y. While Eqs. (2) have been derived for a specific situation, it is readily seen that they have general validity provided the usual sign conventions are made: the positive y axis points from the centroid towards the top of the section; a positive bending moment produces tension at the bottom of the section, and a positive axial force stresses the member in tension. Since $\varepsilon = y\psi$, Eqs. (2) yield

$$\frac{dM}{dN} = -\frac{\varepsilon}{\psi} . \tag{3}$$

A geometric interpretation of Eq. (3) is obtained as follows. Let M_o be the limit moment in pure flexure and N_o the limit force in simple tension. With

$$\mu = M/M_o, \quad \nu = N/N_o, \quad \chi = \psi M_o/N_o \tag{4}$$

Eq. (3) can be written in the form

$$\frac{d\mu}{d\nu} = -\frac{\varepsilon}{\chi} . \tag{5}$$

Using μ and ν as rectangular Cartesian coordinates, represent the fully plastic states of stress by the points of the "yield curve". For a rectangular section, the yield curve is found to consist of the full-line parabolic arcs shown in Fig. 2 (see the paper mentioned in Footnote 3). For an idealized I section with indefinitely thin web and flanges, the yield curve is the broken-line square in Fig. 2; for most structural sections the yield curve falls between the full and the broken lines in Fig. 2. Consider now the fully plastic state of stress represented by the generic point P of the yield curve. If the type of plastic flow which occurs under this state of stress is represented by the "flow vector" with the components ε and χ, Eq. (5) shows that this vector has the direction of the exterior normal of the yield curve at P. On account of this normality the yield curve specifies not only the fully plastic states of stress but also the associated types of plastic flow. In other words, the yield curve represents not only the yield condition but also the flow rule.

In Fig. 3, let the full line represent the yield curve which is supposed to be symmetric with respect to the ν axis. Since the axial forces in structural frames of conventional design play a role secondary to that of the bending moments, only the central part of the yield curve corresponding to, say, $|\nu| < 0.5$ is of importance. For mathematical simplicity, this part may be approximated by two straight line segments AB and BC. This amounts to replacing the actual yield condition by

$$|\mu| = 1 - c|\nu|, \tag{6}$$

where c is a constant. Since the orthogonality between yield curve and flow vector is an essential feature of the general theory of limit design[8], the modified yield condition (6) implies the following rule

$$|\varepsilon| = c|\chi|, \tag{7}$$

the sign of χ and ε agreeing with those of μ and ν, respectively.

The flow rule (7) can be given the following mechanical interpretation. Consider a beam with an off-center hinge under the action of a bending moment M and a compressive force N as shown in Fig. 4. If the distance between the hinge axis and the centroidal axis of the beam equals cM_o/N_o, the rate of bending ψ in the hinge is accompanied by a rate of shortening $\varepsilon = cM_o\psi/N_o = c\chi$ in the centroidal fiber. A hinge at a fixed distance from the centroidal axis therefore enforces the flow rule (7). In view of the fact that M in Fig. 4 is positive while N is negative, the resulting moment of M and N with respect to the hinge axis is

$$|M| + c\,\frac{M_o}{N_o}\,|N| = M_o(|\mu| + c\,|\nu|) = M_o \tag{8}$$

by Eq. (6). Within the limits of validity of the approximate yield condition (6) and the associated flow rule (7), the influence of axial forces on the collapse loads of frames can therefore be evaluated by using off-center hinges that yield under the constant moment M_o. By this device the limit analysis of a frame in which axial forces must be considered can be

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8 W. Prager, "The general theory of limit design", Sectional Address presented at the 8th International Congress of Theoretical and Applied Mechanics, Istanbul, 1952, to appear in the Proceedings of the Congress.

reduced to the limit analysis of a frame in which axial forces may be neglected.

For an idealized I section with the broken-line yield square of Fig. 2, the constant c equals unity. Moreover, for such a beam $M_o/N_o = H/2$, where H is the depth of the beam. The distance of the yield hinge from the centroidal axis therefore equals H/2 in this case. For actual structural I sections this distance is smaller than one half of the depth of the section. The fact that the yield hinges used in the analysis in which axial forces are neglected must only be slightly displaced to include the influence of axial forces affords a valuable intuitive insight into the role of axial forces in the plastic collapse of frames.

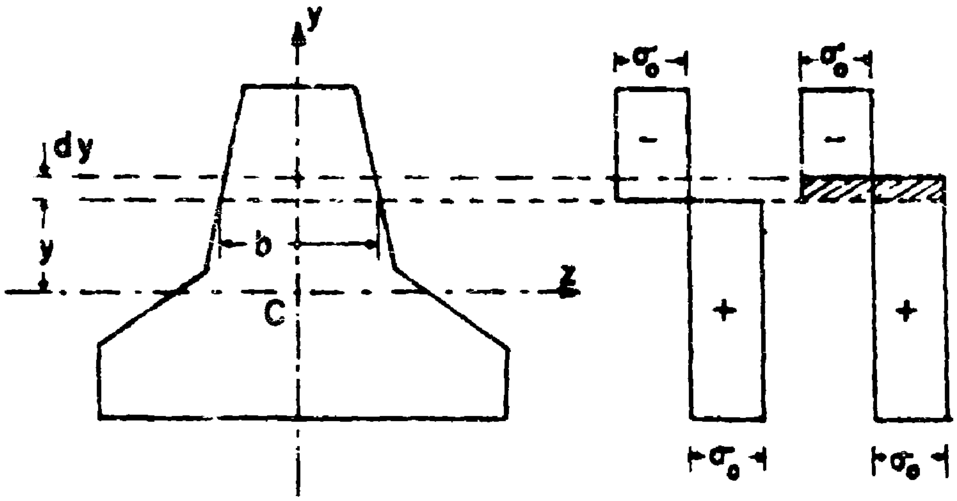
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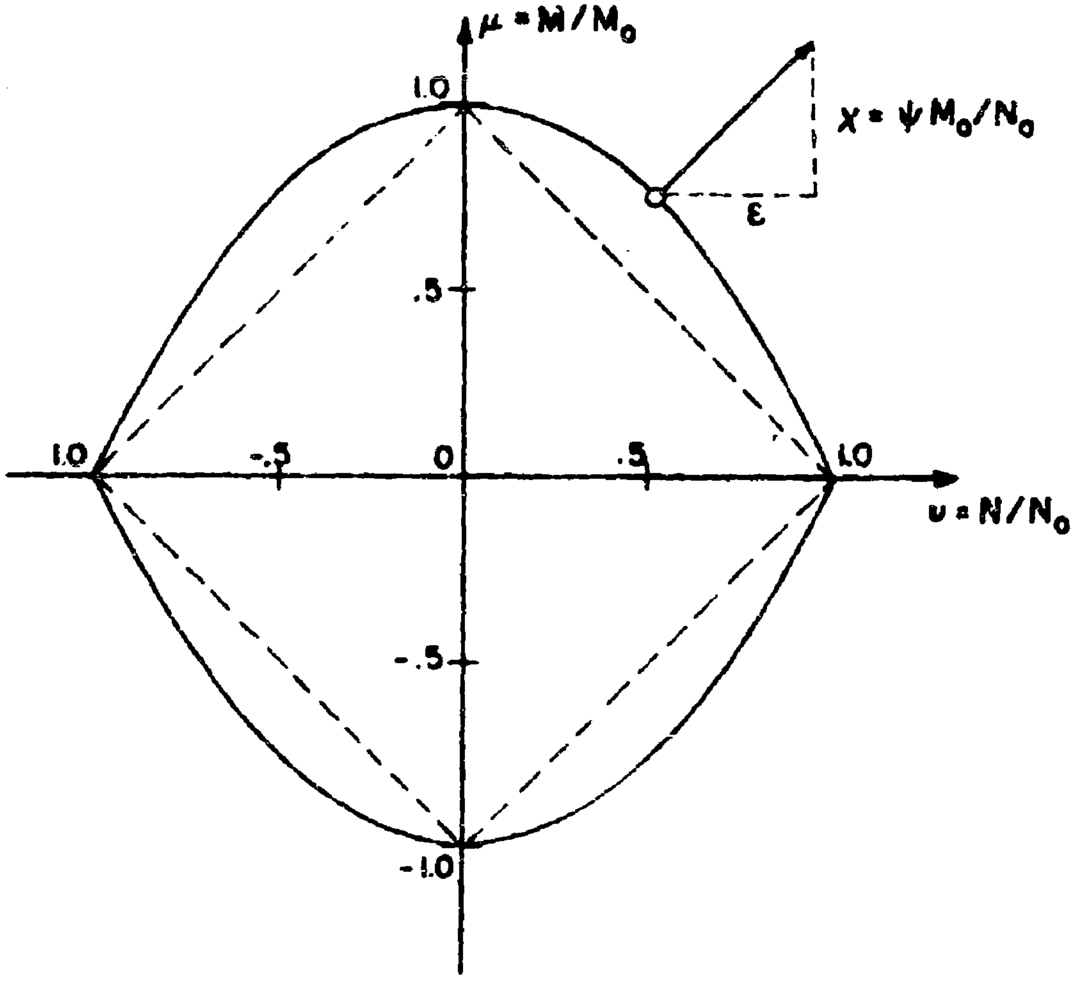
Fig. 1


$\mu = M/M_0$
$\chi = \psi M_0/N_0$
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-1.0
1.0
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0
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1.0
$\nu = N/N_0$

Fig. 2

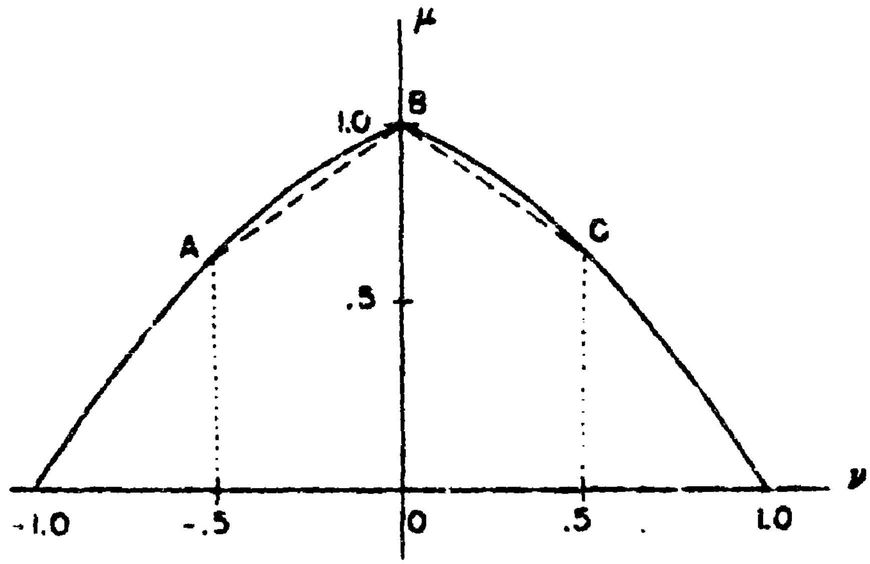

Fig. 3

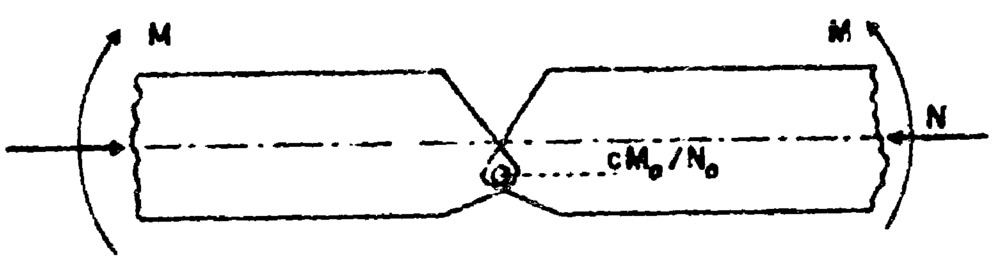

Fig. 4

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